Universal Properties
A categorical look at undergraduate algebra and topology

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Category Theory

- Maths is Abstraction
- Category Theory: more abstraction

Universal Properties

- Within one category
- Mixing categories
1 Category Theory
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- Category Theory: more abstraction

2 Universal Properties
- Within one category
- Mixing categories
What is Abstraction?

Abstraction

- Take example/situation/idea.
- Determine some (important) properties.
- “Lift” those away from the example/situation/idea.
- Work with abstracted properties.
- Should get many more examples which also fit these “lifted” properties.

Examples

- My pet and my friend’s pet are both cats.
- Cats, dogs, dolphins are all mammals.
- My home, my old school, the maths department are all buildings.
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Examples

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The probably most important step of abstraction in the history of mathematics:

- “3 apples” $\rightarrow$ “3”
Numbers

The probably most important step of abstraction in the history of mathematics:

- “3 apples” → “3”

After that also (not necessarily in this order)

- negative numbers (abstraction of debt?)
- rational numbers (abstraction of proportions)
- real numbers (abstraction of lengths)
More examples

Groups

- Addition in $\mathbb{Z}$, “clock” addition (mod $n$) and composing symmetries have similar properties.
- Isolate the properties.
- Define an abstract group.
- Get lots more examples, and a whole area of mathematics.
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Equivalence relations
- Study equality, congruence (mod $n$) and “having same image under a function”.
- Isolate: reflexivity, symmetry, transitivity.
- Define equivalence relation.
- Work with the abstract idea rather than one example .....
We notice throughout our studies that certain objects come with special maps:

<table>
<thead>
<tr>
<th>objects</th>
<th>“structure preserving” maps</th>
</tr>
</thead>
<tbody>
<tr>
<td>sets</td>
<td>functions</td>
</tr>
<tr>
<td>groups</td>
<td>group homomorphisms</td>
</tr>
<tr>
<td>rings</td>
<td>ring homomorphisms</td>
</tr>
<tr>
<td>modules/vector spaces</td>
<td>linear maps</td>
</tr>
<tr>
<td>topological spaces</td>
<td>continuous maps</td>
</tr>
</tbody>
</table>
One more level of abstraction

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A \rightarrow B \rightarrow C
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- There is an identity:

\[ A \xrightarrow{1_A} A \xrightarrow{f} B = A \xrightarrow{f} B = A \xrightarrow{f} B \xrightarrow{1_B} B \]
One more level of abstraction

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\]

- There is an identity:

- Composition is associative: \((h \circ g) \circ f = h \circ (g \circ f)\)

\[
A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D
\]
Definition of a category

A category $C$ consists of

- a collection $\text{ob}C$ of objects $A, B, C, \ldots$ and
- for each pair of objects $A, B \in \text{ob}C$, a collection $C(A, B) = \text{Hom}_C(A, B)$ of morphisms $f : A \rightarrow B$, equipped with
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equipped with

- for each $A \in \text{ob}C$, a morphism $1_A : A \rightarrow A$, the identity,
- for each triple $A, B, C \in \text{ob}C$, a composition

\[ \circ : \text{Hom}(A, B) \times \text{Hom}(B, C) \rightarrow \text{Hom}(A, C) \]

\[ (f, g) \mapsto g \circ f \]

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such that the following axioms hold:

1. **Identity:** For $f: A \rightarrow B$ we have $f \circ 1_A = f = 1_B \circ f$.

2. **Associativity:** For $f: A \rightarrow B$, $g: B \rightarrow C$ and $h: C \rightarrow D$ we have $h \circ (g \circ f) = (h \circ g) \circ f$. 
What is Category Theory?

- One more level of abstraction.
  - addition and symmetries of polyhedra $\rightarrow$ groups
  - equality and congruence $\rightarrow$ equivalence relations
  - integers $\rightarrow$ ring theory

Category Theory is “mathematics about mathematics”.
- sets, groups, vectorspaces etc. $\rightarrow$ categories

- A language for mathematicians.
- A way of thinking.
In category theory:

We are not only interested in objects (such as sets, groups, ...), but how different objects of the same kind *relate* to each other. We are interested in global structures and connections.
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### Motto of category theory

We want to really understand how and why things work, so that we can present them in a way which makes everything “look obvious”.

Examples of categories

- Any collection of sets with a certain structure and structure-preserving maps will form a category.

But also:

A group $G$ is a one-object category with the group elements as morphisms:
- $e \in G$ is identity morphism.
- Group multiplication is composition.

A poset $P$ is a category:
- The elements of $P$ are the objects.
- $\text{Hom}(x, y)$ has one element if $x \leq y$, empty otherwise.
- Reflexivity gives identities.
- Transitivity gives composition.
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Universal Property Template

Template

$X$ has property $\mathcal{P}$, and if $Y$ also has property $\mathcal{P}$, then there is a unique map between $X$ and $Y$ which “fits with the property $\mathcal{P}$”.
Universal Property Template

Template

\[ X \text{ has property } P, \text{ and if } Y \text{ also has property } P, \text{ then there is a unique map between } X \text{ and } Y \text{ which “fits with the property } P”. \]

Note: could be unique map \( X \rightarrow Y \) or \( Y \rightarrow X \).
Terminal objects

First example: property $\mathcal{P} = \text{"is an object" ("empty property")}.}$
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**Definition**

An object $T \in \text{ob} \mathcal{C}$ is called **terminal object** when there is, for every $A \in \text{ob} \mathcal{C}$, a unique morphism $A \longrightarrow T$ in $\mathcal{C}$. 

Examples 

- **Sets** $X$: exactly one function $X \longrightarrow \{\ast\}$.
- **Groups** $G$: exactly one group hom $G \longrightarrow 0 = \{e\}$.
- **Vector spaces** $V$: exactly one linear map $V \longrightarrow 0$.
- **Top. spaces** $X$: exactly one continuous map $X \longrightarrow \{\ast\}$. 

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Terminal objects

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Initial objects

\( \mathcal{P} = \text{"is an object"}, \) but \textit{unique arrow from} rather than \textit{unique arrow to}.

**Definition**

An object \( I \in \text{ob}\mathcal{C} \) is called \textbf{initial object} when there is, for every \( A \in \text{ob}\mathcal{C} \), a unique morphism \( I \rightarrow A \) in the category \( \mathcal{C} \).
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- Rings \( R \): exactly one ring homomorphism \( \mathbb{Z} \rightarrow R \).
- Sets \( X \): exactly one function \( \emptyset \rightarrow X \).
- Topological spaces: also \( \emptyset \).
Universal property of a product

\[ A \times B \]

\[ \pi_1 \quad \pi_2 \]

\[ A \\ \rightarrow \quad B \]

\[ \exists ! h \text{ which satisfies } \pi_1 \circ h = f \text{ and } \pi_2 \circ h = g. \]

Examples
- Sets: cartesian product \( A \times B = \{(a, b) | a \in A, b \in B\} \).
- Groups: cartesian product with pointwise group structure.
- Topological spaces: cartesian product with the product topology.
Universal property of a product

\[ C \xrightarrow{\exists! h} A \times B \]

\[ A \xleftarrow{\pi_1} \]  \[ f \rightarrow \]  \[ \pi_2 \rightarrow B \]

\[ g \rightarrow \]  \[ \rightarrow C \]

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$\exists! h$ which satisfies $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$.

Examples

- Sets: cartesian product $A \times B = \{(a, b) \mid a \in A, b \in B\}$.
- Groups: cartesian product with pointwise group structure.
- Topological spaces:
Universal property of a product

There exists a unique morphism $h : C \to A \times B$ such that $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$.

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Universal property of a coproduct

\[ A \xleftarrow{\iota_1} A + B \xrightarrow{\iota_2} B \]

Examples:
- Disjoint union of sets
- Disjoint union of topological spaces
- Free product of groups
- (External) direct sum of modules
  \[ M \oplus N = M \times N \]
Coproducts

Universal property of a coproduct

∃! h s.t. \( h \circ \iota_1 = f \), \( h \circ \iota_2 = g \)
Coproducts

Universal property of a coproduct

\[ A \overset{\iota_1}{\longrightarrow} A + B \overset{!}{\longrightarrow} C \overset{\iota_2}{\longleftarrow} B \]

\[ \exists ! h \text{ s.t. } h \circ \iota_1 = f, \ h \circ \iota_2 = g \]

Examples

- disjoint union of sets \( A \coprod B \).
- disjoint union of topological spaces.
- free product of groups \( G \ast H \).
- (external) direct sum of modules \( M \oplus N = M \times N \).
A stranger example

Poset as category: \( \text{Hom}(x, y) \) has one element if \( x \leq y \), empty otherwise.

**Universal properties in a poset**

- Terminal object is “top element” (if it exists).
- Initial object is “bottom element” (if it exists).
- Products are meets (e.g. in a powerset: intersection).
- Coproducts are joins (e.g. in a powerset: union).
Any universal object is unique (up to iso)

- Suppose $X$, $Y$ both universal for $P$. 

Then $g \circ f : X \to X$ also "commutes with $P".

Identity 1 $X : X \to X$ always "commutes with $P".

But have unique such: so $1_X = g \circ f$.

Similarly $1_Y = f \circ g$.

So $X \cong Y$. 

Any universal object is unique (up to iso)

- Suppose $X, Y$ both universal for $\mathcal{P}$.
- $\exists$ unique $f : X \rightarrow Y$ and $g : Y \rightarrow X$ “commuting with $\mathcal{P}$”.

So $X \sim Y$. 

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Uniqueness

Any universal object is unique (up to iso)

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- Then $g \circ f: X \rightarrow X$ also “commutes with $\mathcal{P}$”.

**Identity 1**: $X \rightarrow X$ always “commutes with $\mathcal{P}$”. But have unique such: so $1_X = g \circ f$.

**Identity 2**: $1_Y = f \circ g$. So $X \cong = Y$. 

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- Similarly $1_Y = f \circ g$.
- So $X \cong Y$. 
Turning around arrows

Initial is “opposite” of terminal

- Terminal $T$: for all $A$, $\exists!$ map $A \rightarrow T$.
- Initial $I$: for all $A$, $\exists!$ map $A \leftarrow I$. 
Turning around arrows

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- Initial $I$: for all $A$, $\exists!$ map $A \leftarrow I$.

Coproduct is “opposite” of product
Zero objects

- For groups and modules, initial = terminal.
- Define zero-object 0 to be both initial and terminal.
- Gives at least one map between any two objects:

\[ A \rightarrow 0 \rightarrow B \]
Coinciding properties

### Zero objects
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- Define zero-object 0 to be both initial and terminal.
- Gives at least one map between any two objects:

\[ A \rightarrow 0 \rightarrow B \]

### Direct products
- **Direct product** is both product and coproduct.
- E.g. direct sum of modules (vector spaces, abelian groups...
Kernels

Universal property of a kernel

\[
\begin{align*}
K & \xrightarrow{k} A \xrightarrow{f} B \\
C & \xrightarrow{\exists! l} A
\end{align*}
\]

Kernel of \( f \) is universal map whose post-composition with \( f \) is zero.
Kernel of $f$ is universal map whose post-composition with $f$ is zero.

In terms of elements

$$K = \{ k \in A \mid f(k) = 0 \}$$

$k$ the inclusion into $A$. 

Universal property of a kernel
Cokernels: “turn around the arrows”

Universal property of a cokernel

Cokernel of $f$ is universal map whose pre-composition with $f$ is zero.
Cokernels: “turn around the arrows”

Universal property of a cokernel

\[ A \xrightarrow{f} B \xrightarrow{q} Q \]

\[ 0 \xrightarrow{g} \exists! p \]

Cokernel of \( f \) is universal map whose pre-composition with \( f \) is zero.

In modules/vector spaces/abelian groups

\[ Q = B / \text{Im}(f) = \{ b + \text{Im}(f) \} , \quad q \text{ the quotient map.} \]

\[ A \xrightarrow{} \text{Im}(f) \xrightarrow{} B \xrightarrow{} B / \text{Im}(f) \]
Tensor Product of Vector Spaces/Modules

\[ V \times W \xrightarrow{\varphi} V \otimes W \]

\[ h \downarrow \exists! \overline{h} \]

\( \varphi \) is universal bilinear map out of \( V \times W \), tensor product \( U \otimes V \) “makes bilinear \( h \) into linear \( \overline{h} \)”. 
Tensor Product of Vector Spaces/Modules

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Construction

- Actual construction is complicated and slightly tedious.
- Working with universal property is often easier than with the elements.
Abelianisation of a group

\[ G \rightarrow \text{ab} \ G \rightarrow A \]

Every group hom to an abelian group \( A \) factors uniquely through the abelianisation.
Abelianisation

Abelianisation of a group

\[ G \xrightarrow{} ab\ G \]

\[ \exists! \quad \downarrow \]

\[ A \]

Every group hom to an abelian group \( A \) factors uniquely through the abelianisation.

Construction

- \( ab\ G = G/[G, G] \)
- \([G, G]\) is {\textbf{commutator}}: normal subgroup generated by all \( aba^{-1}b^{-1} \).
Every injective ring hom to a field $K$ factors uniquely through the field of fractions.

“Smallest field into which $R$ can be embedded.”
Field of fractions

Field of fractions of an integral domain

Every injective ring hom to a field $K$ factors uniquely through the field of fractions.
“Smallest field into which $R$ can be embedded.”

Construction

- $F = \{(a, b) \in R \times R \mid b \neq 0\}/\sim$
- equivalence relation $\sim$ is $(a, b) \sim (c, d)$ iff $ad = bc$. 
Compactification of a topological space

Every continuous map to a compact Hausdorff space $K$ factors uniquely through the Stone-Čech compactification.
Stone-Čech Compactification

Compactification of a topological space

\[ X \rightarrow \beta X \rightarrow K \]

Every continuous map to a compact Hausdorff space \( K \) factors uniquely through the Stone-Čech compactification.

Generalisation

Abelianisation and Stone-Čech compactification are examples of adjunctions: very important concept in Category Theory.
Advantages of Universal Properties

- **Tidyier**: details may be messy, working with universal property can give clear and elegant proofs.
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- **Functorial**: defining things via universal properties gives them good categorical properties (used all over maths).
Why bother?

Advantages of Universal Properties

- **Tidyier**: details may be messy, working with universal property can give clear and elegant proofs.
- **Transferable**: situations with different details may have same universal property: transfer ideas/proofs/...
- **Functorial**: defining things via universal properties gives them good categorical properties (used all over maths).
- **Useful**: e.g. to show two objects are isomorphic, show they satisfy same universal property.
Thanks for listening!